Efficient decoding of random errors for quantum expander codes

Antoine Grospellier, Anthony Leverrier, Omar Fawzi

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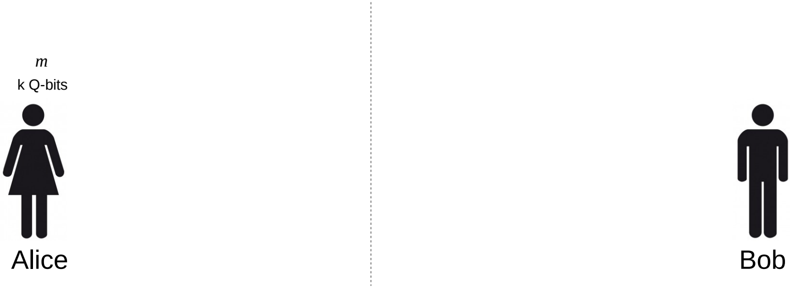
10 Novembre 2017

Alice

Bob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k Q-bits
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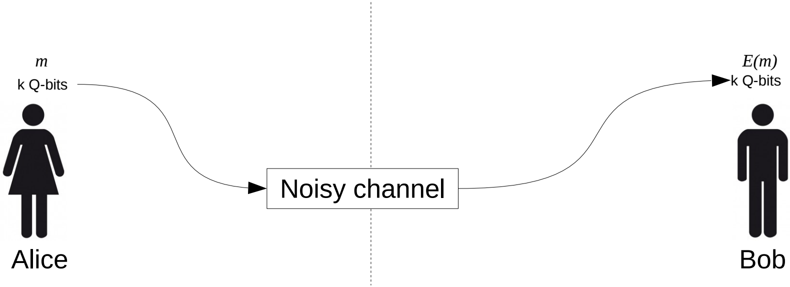
m

k Q-bits

Alice

Noisy channel

Bob



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$E(m)$
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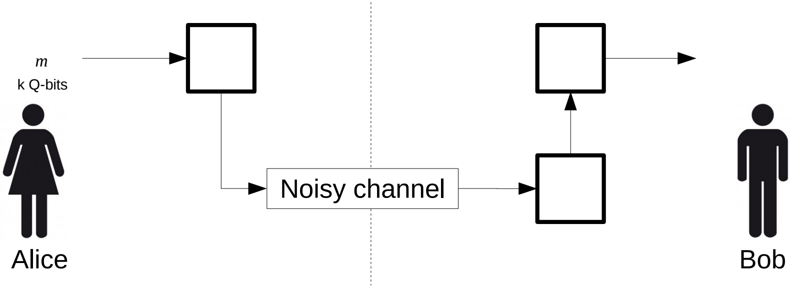
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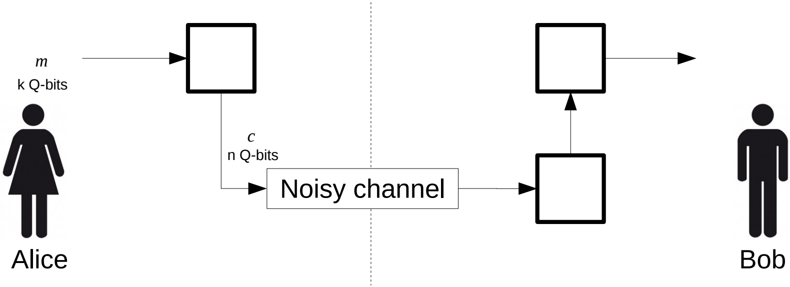

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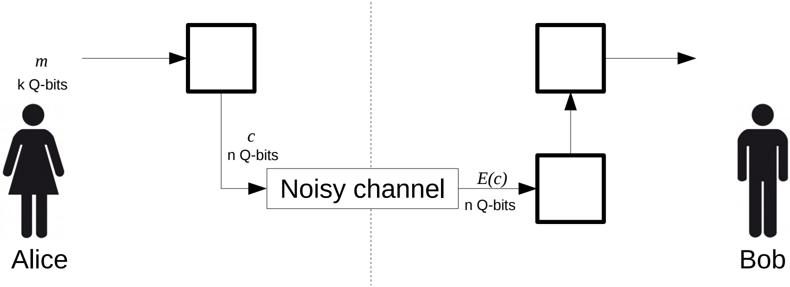
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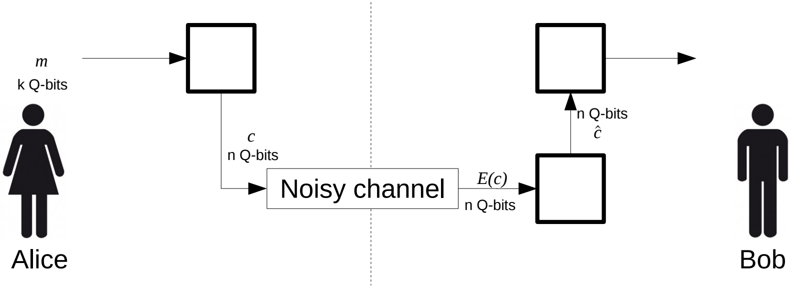

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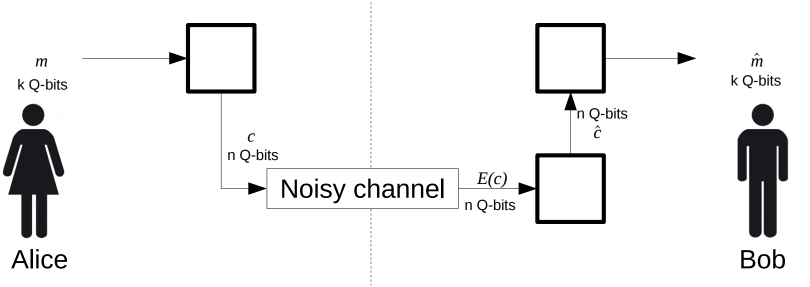
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Content of the talk

The hypergraph product of an expander code :

- is an LDPC quantum code
- has a constant rate
- has a minimal distance : $d = \Theta(\sqrt{n})$

The decoding algorithm :

- has a capacity of correction : $\Theta(\sqrt{n})$
- **corrects the error with high probability for the depolarizing channel**

Content of the talk

1. Classical expander codes
2. Quantum expander codes
3. Our contribution

Plan

1. Classical expander codes
2. Quantum expander codes
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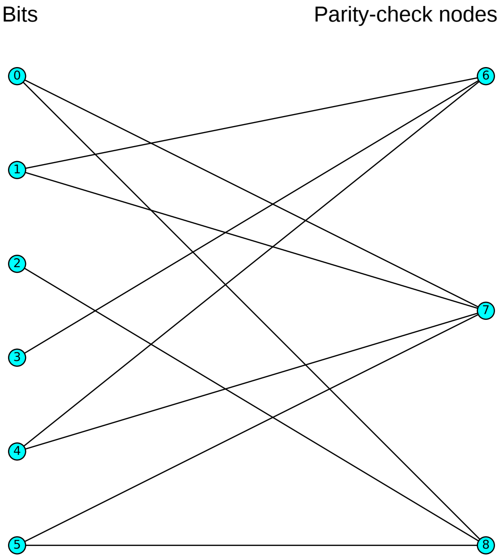
Tanner graph of a code



Classical expander codes [Sipser & Spielman, '96]

Theorem [Sipser & Spielman, '96]

We can construct a good family $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of $[n, k, d]$-error correcting codes. Here "Good" means :

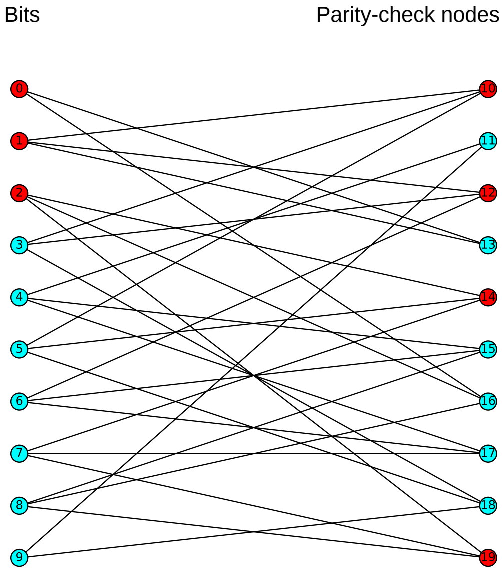
- This family is LDPC
- k and d are linear in n
- There exists an efficient correcting algorithm

Remark

Good expander codes can be found efficiently by picking a random biregular graph

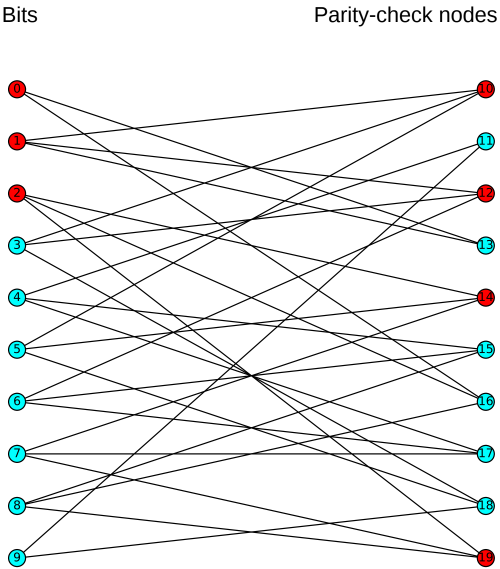
Decoding algorithm

- Error : $e_0 = \{0, 1, 2\}$
- Unsatisfied check-nodes (syndrome) : $\{10, 12, 14, 19\}$
- Satisfied check-nodes : $\{11, 13, 15, 16, 17, 18\}$

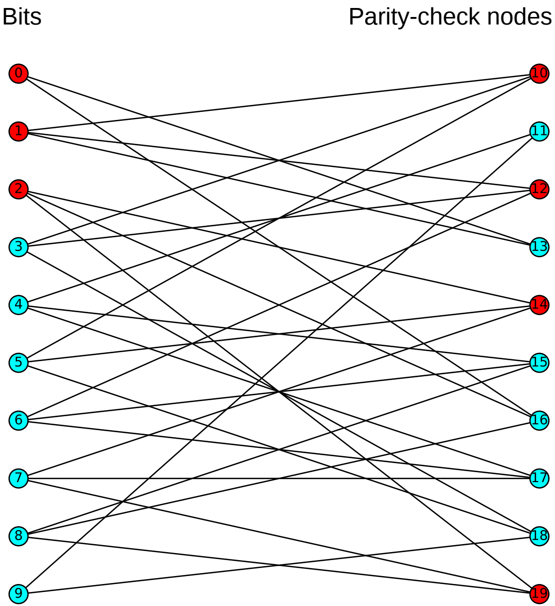


Decoding algorithm

- INPUT : syndrome
- The error e_0 is unknown
- OUTPUT : e a set of bits
- GOAL : $e = e_0$
- The algorithm flips a bit when it decreases the syndrome



Decoding algorithm : first example



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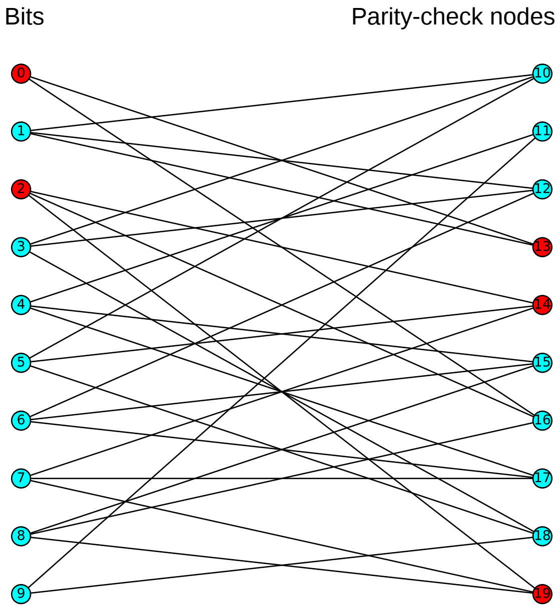

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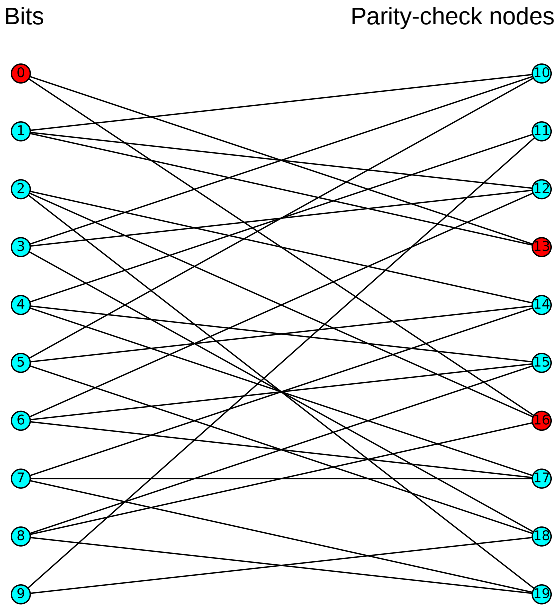

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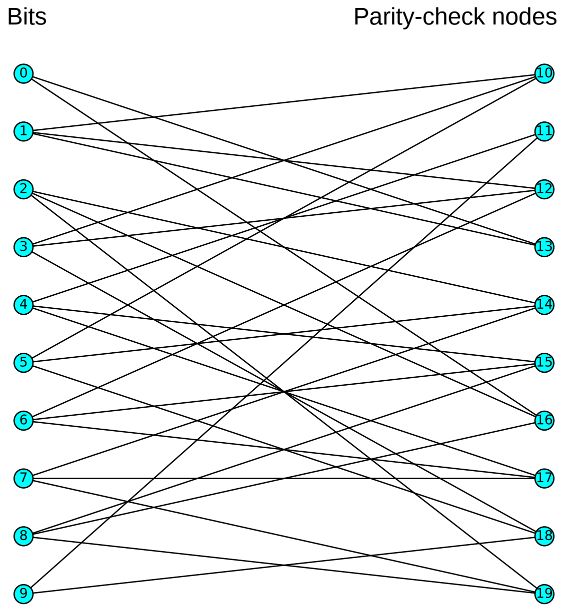
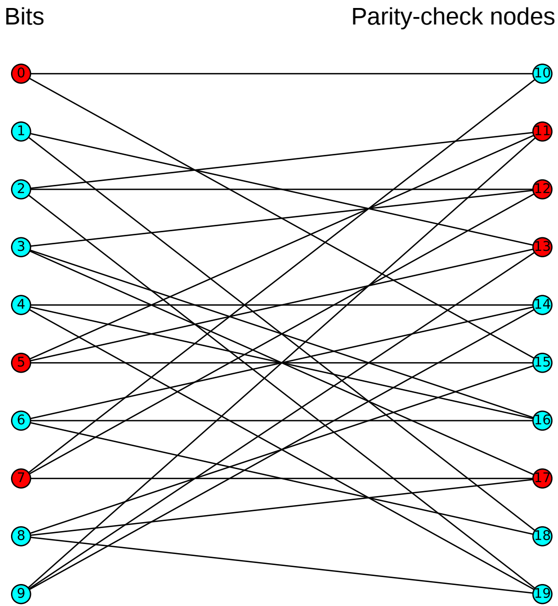


Decoding algorithm : second example



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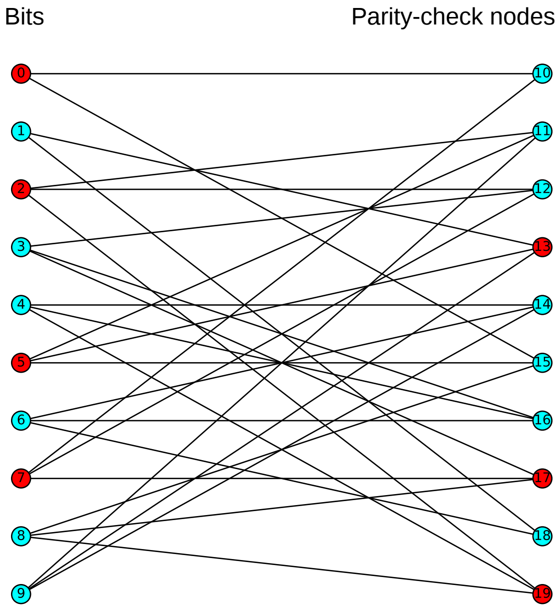

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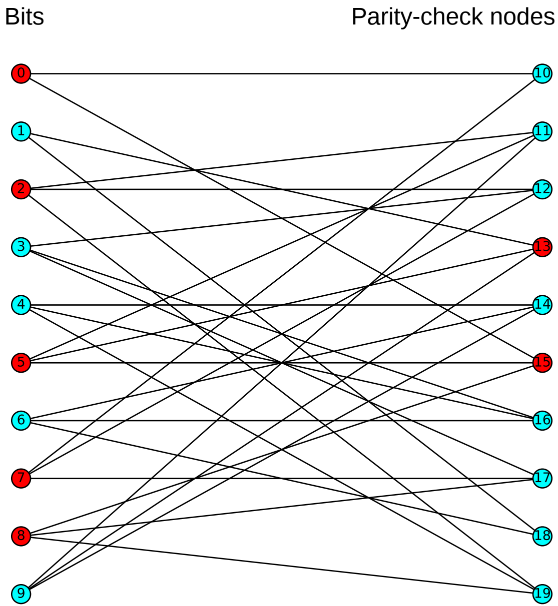

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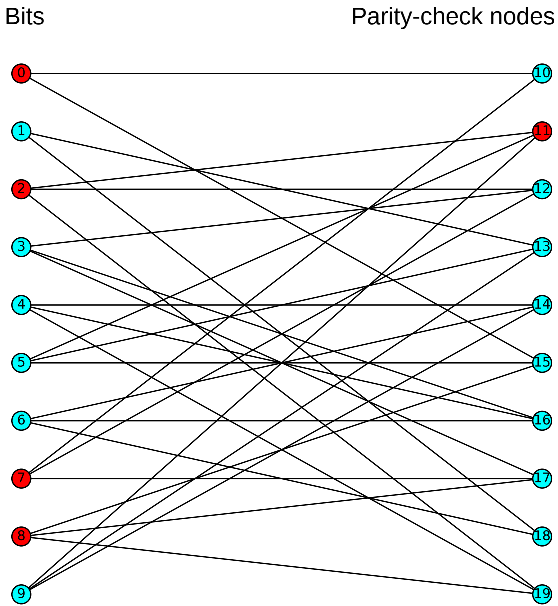

Plan

1. Classical expander codes
2. Quantum expander codes
3. Our contribution

From classical to quantum error correcting codes

Hypergraph product [Tillich & Zémor, '09]

Using a classical code $\mathcal{C}$, we can construct a $[\![n, k, d]\!]$-CSS code with :

- $k = \Theta(n)$
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Theorem [Leverrier & Tillich & Zémor, '15]

For the hypergraph product of an expander code ($\epsilon < 1/6$) :
There is an efficient decoding algorithm for this code.
This algorithm corrects any error of size $\leq \Theta(\sqrt{n})$
This algorithm is very close to the algorithm of Sipser and Spielman

Plan

Our work

- **Question :** What happens for random errors of size $\Theta(n)$?
- **Depolarizing channel :** each Q-bit has an X-type error (resp. Y,Z-type error) with probabilty p independently

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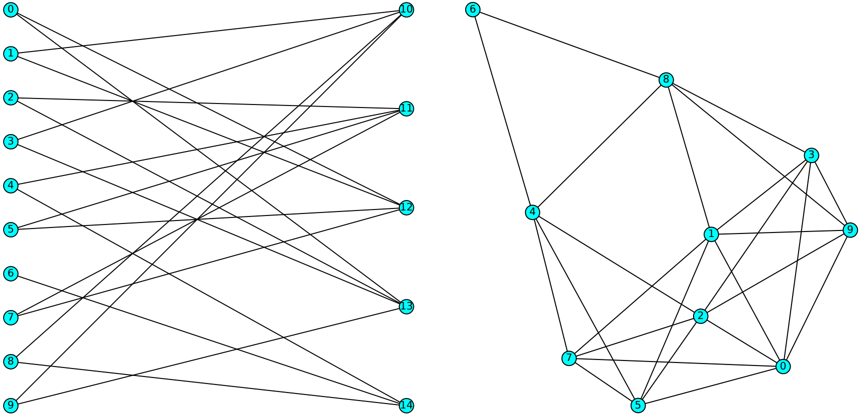
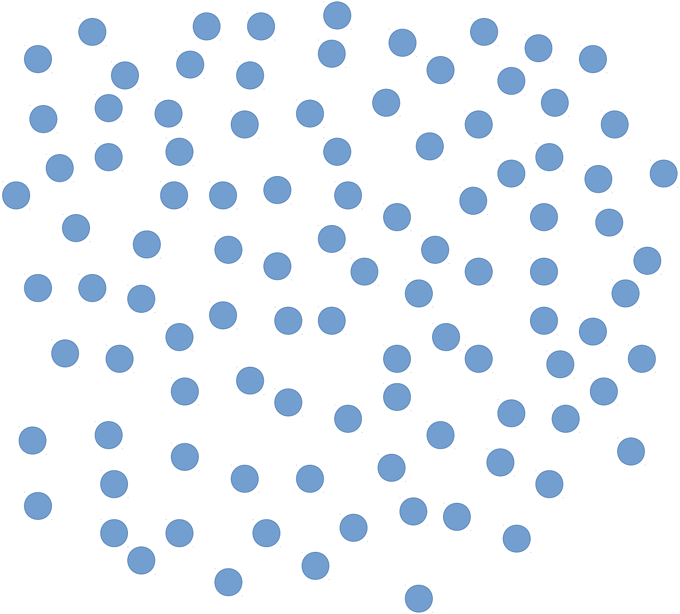
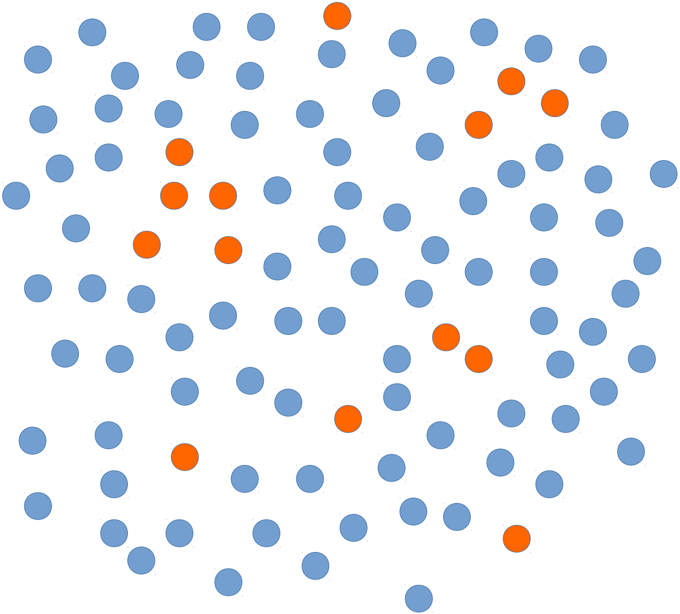
Idea : The algorithm is local :

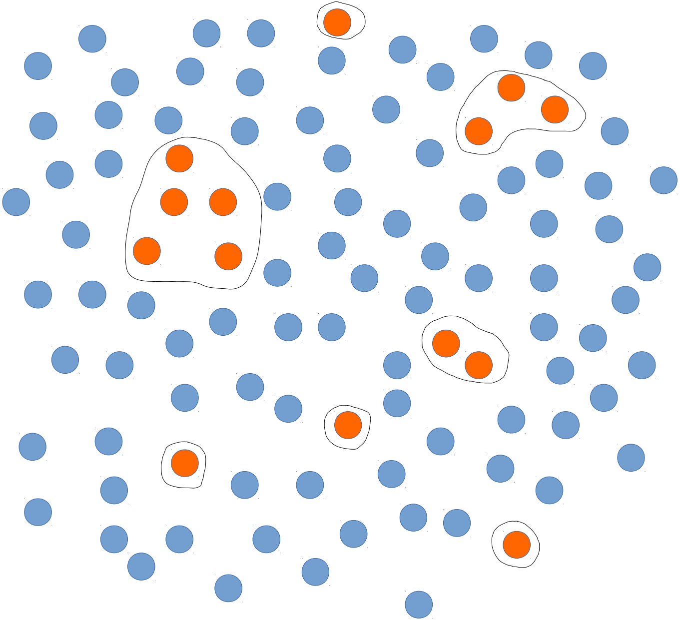
- If two errors are far they don't interact
- The initial error can be decomposed in clusters of size $O(ln(n))$
- If there is no cluster of size $\Theta(\sqrt{n})$ during the algorithm, the error will be corrected

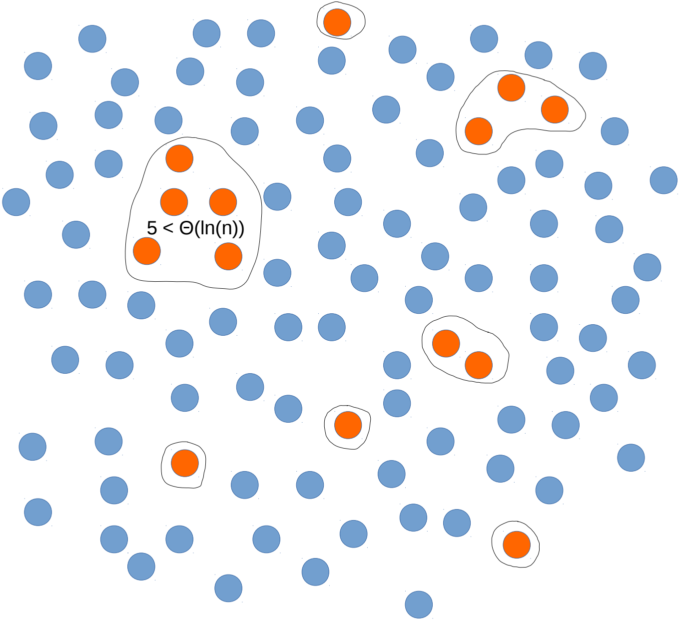
Locality of the algorithm : the adjacency graph







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Percolation Theorem

For a probability of error $p < \frac{1}{d-1}$, whp :

- The size of any connected components is $\leq \Theta(ln(n))$

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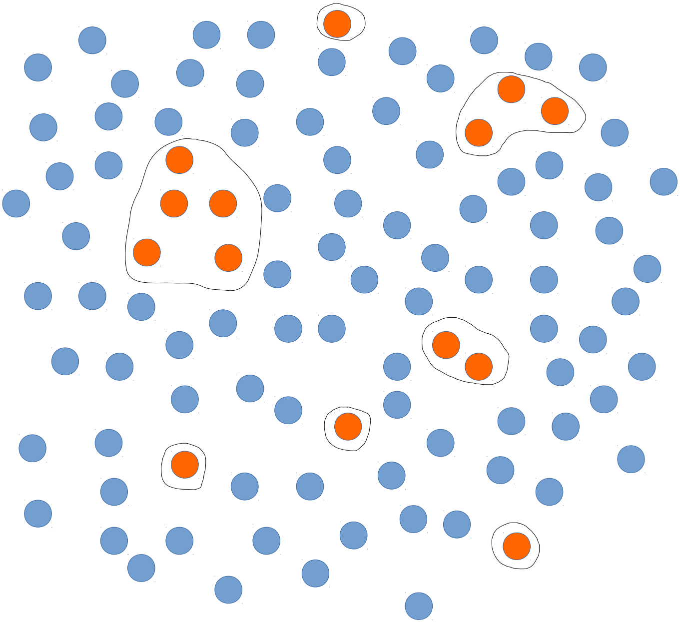
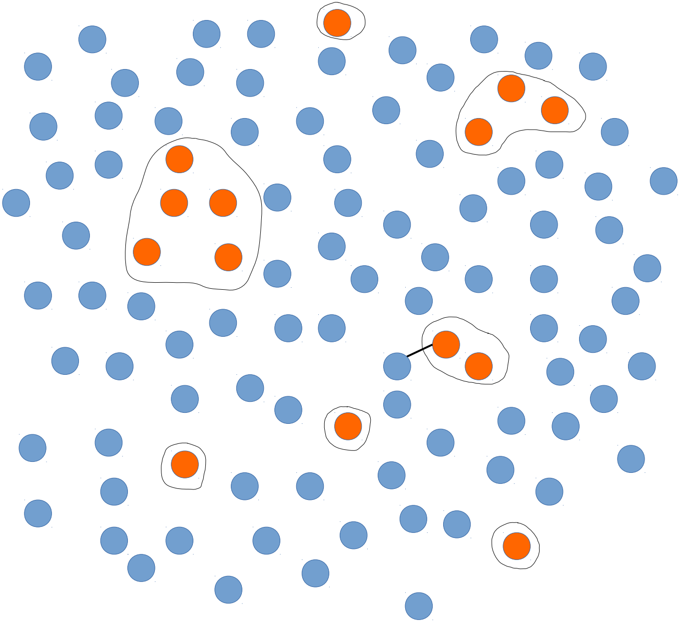
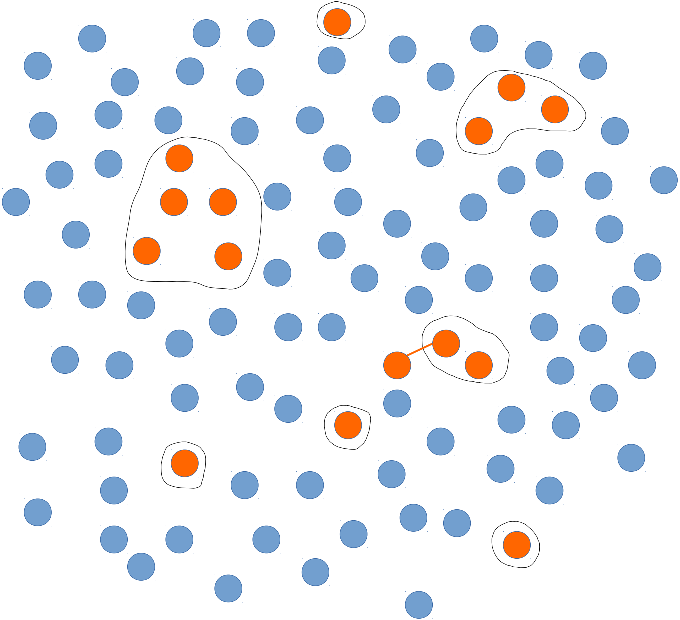
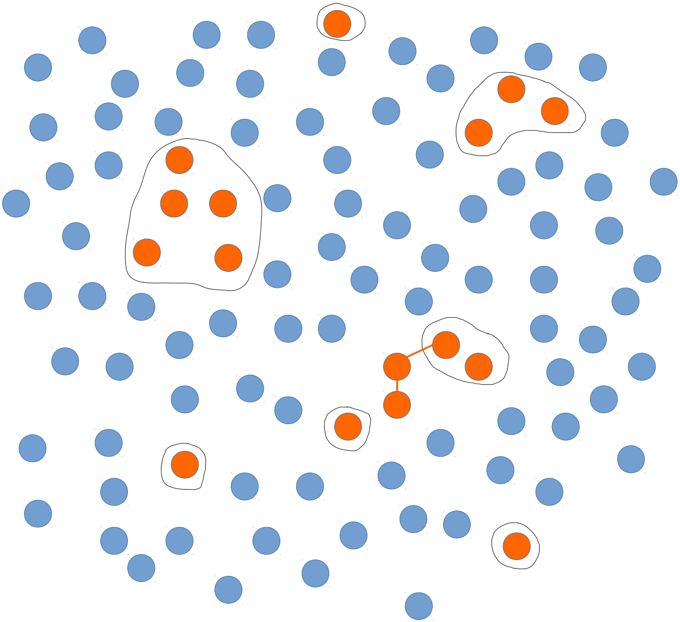
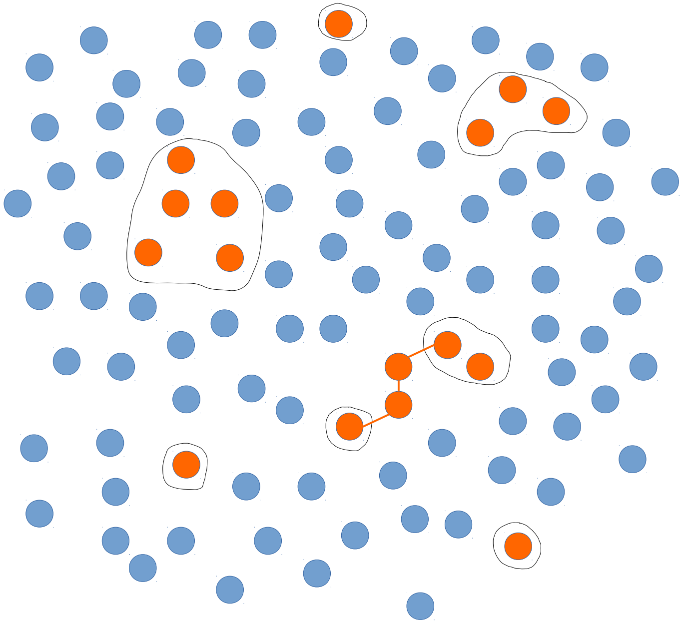
Problem : Some clusters can merge during the decoding











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$\forall \alpha > 0$, if $p < cst(\alpha, d)$, whp :

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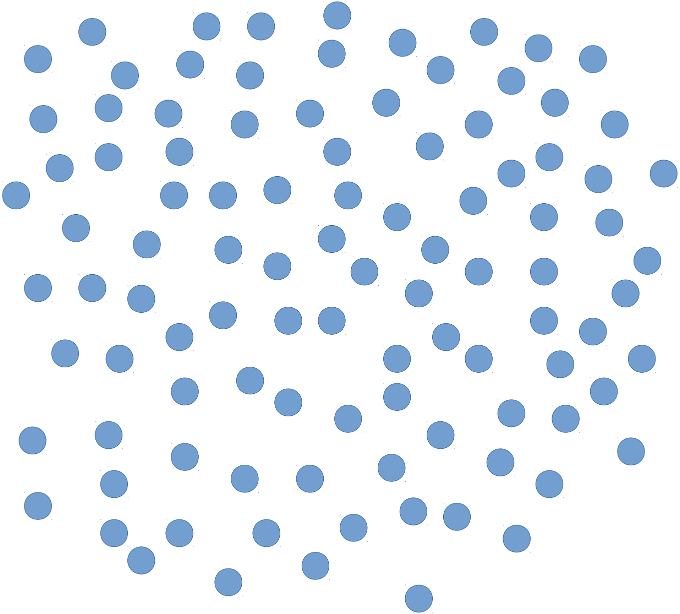
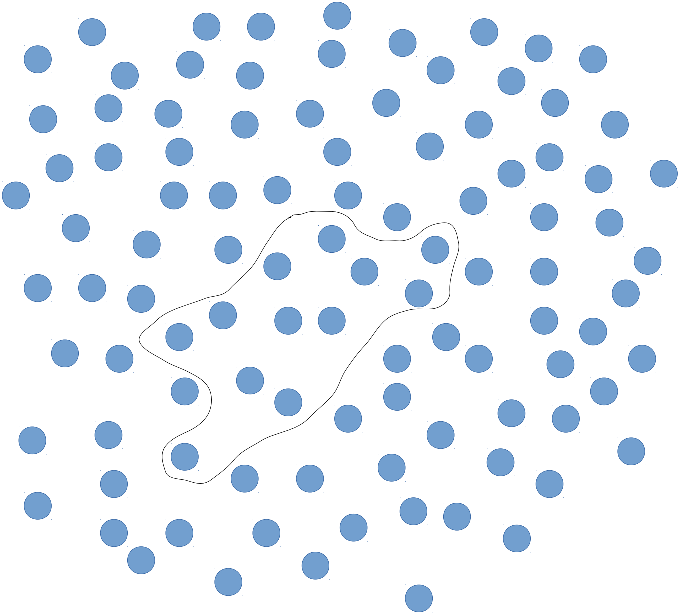
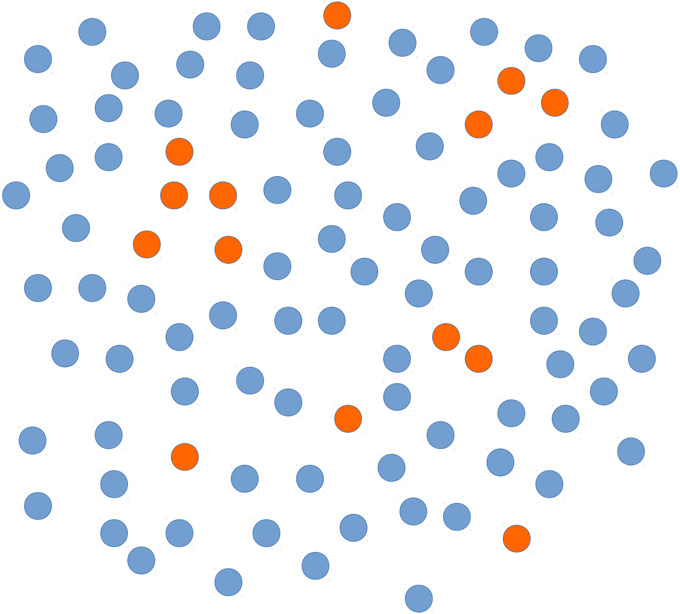
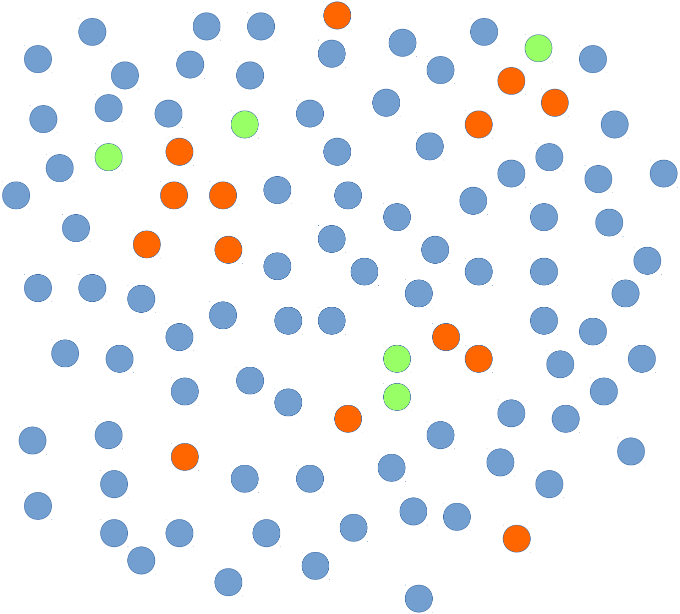
Complexity of the decoding algorithm

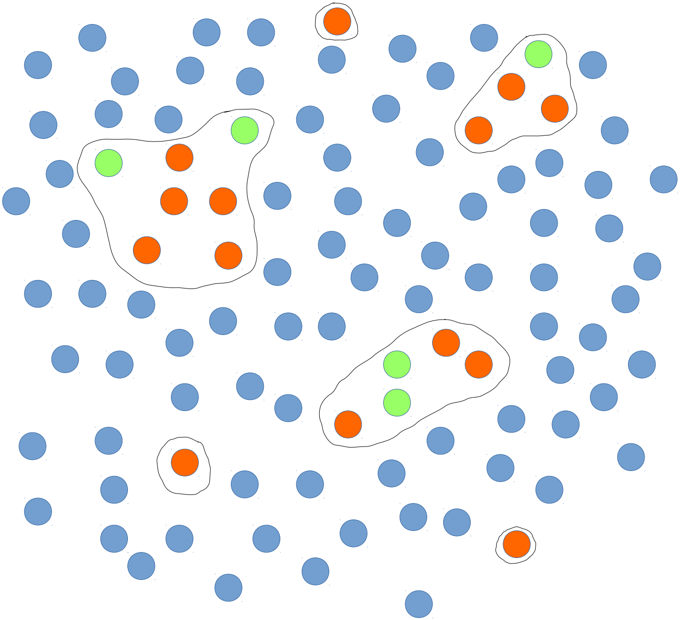
The number of flips is linear in the size of the initial error

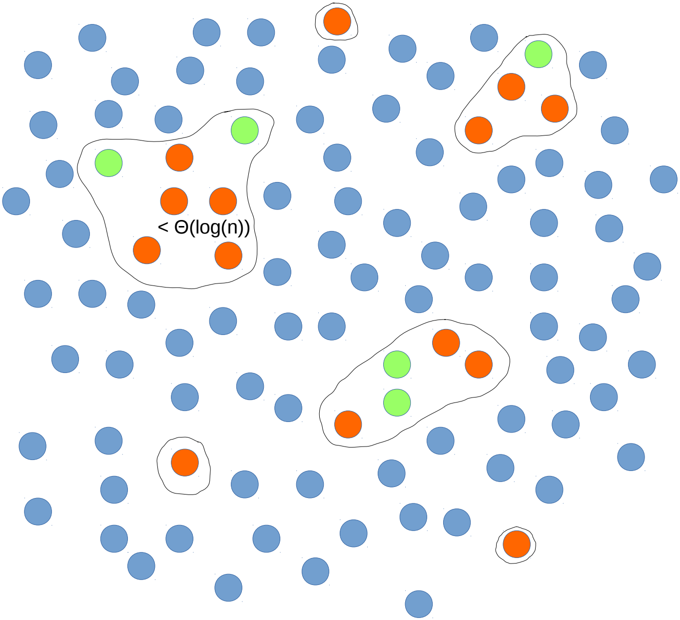








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Ideas to improve this bound :

1. Improve the bound in the percolation theorem :
   What is the critical probability ?
2. Restrict the proof to interesting clusters :
   * The diameter of an interesting cluster is $\leq \Theta(ln(ln(n)))$
   * The number of edges inside an interesting cluster is large (ideas from bootstrap percolation)

Conclusion

The hypergraph product of an expander code :

- is an LDPC quantum code
- has a constant rate
- has a minimal distance : $d = \Theta(\sqrt{n})$

The decoding algorithm :

- has a capacity of correction : $\Theta(\sqrt{n})$
- corrects the error with high probability for the depolarizing channel

Futur work :

- improve our bound
- run simulations
- apply this result to fault tolerant quantum computation (Gottesman)

Thank you for your attention

A motivation : fault-tolerant quantum computation

Threshold Theorem [Ben-Or & Aharonov, '97]

We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **quasi-linear**

Theorem : what we proved

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What we hope to prove using [Gottesman, '13]

We can simulate a quantum circuit with perfect gates by a circuit with noisy gates of size **linear**

$n = 10, m = 5, d_1 = 2, d_2 = \frac{n \times d_1}{m} = 4$

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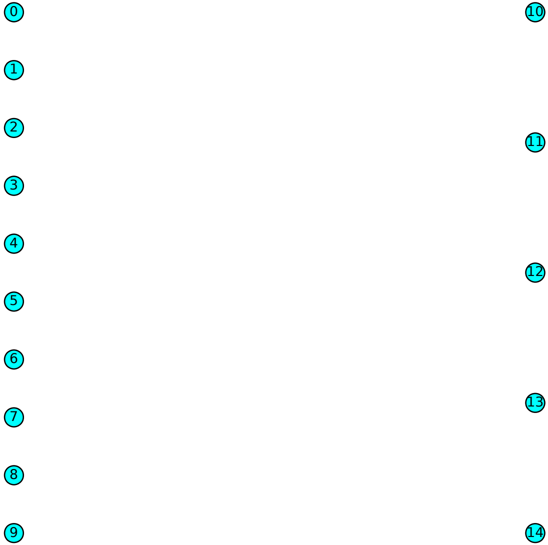

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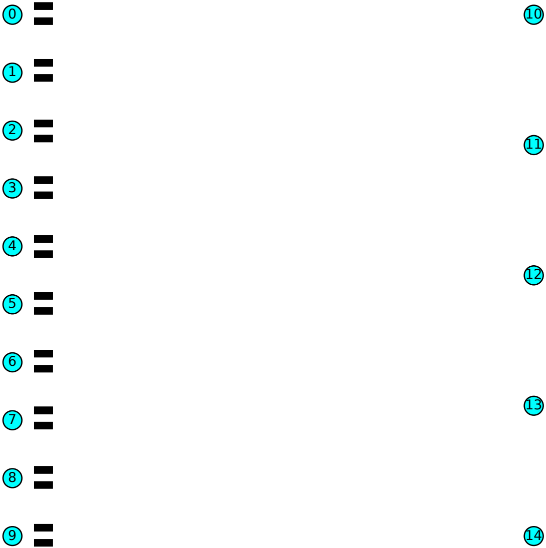

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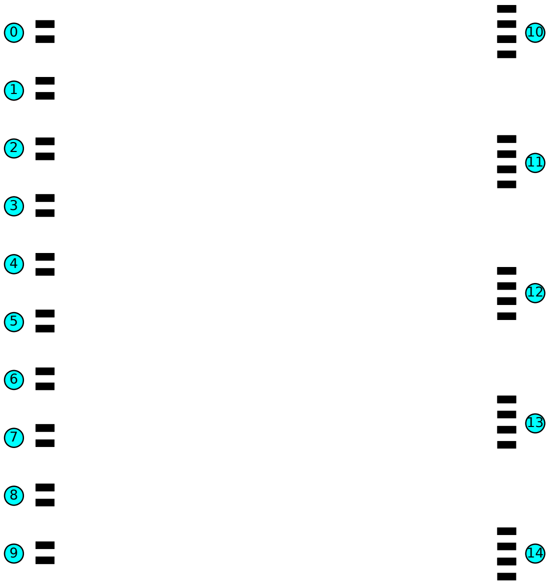

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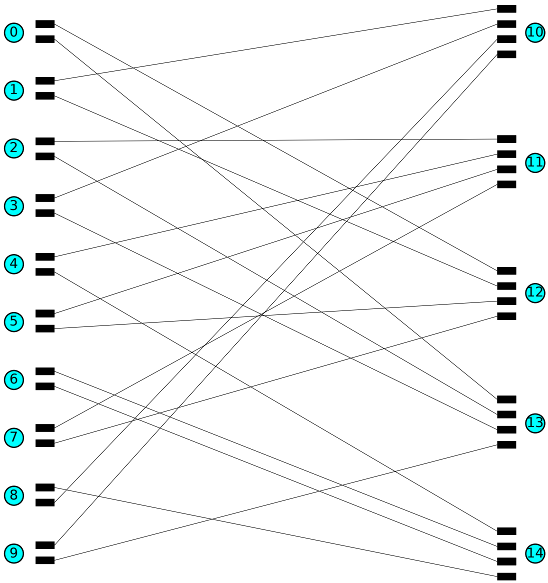

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